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On the geometry of coset models with flux

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Abstract. We study the 3-form flux $H_{\mu\nu\lambda}$ associated with the semi-classical geometry of G/H gauged Wess–Zumino–Witten models. We derive a simple, general expression for the flux in an orthonormal frame and use it to explicitly verify conformal invariance to the leading order in α'. For supersymmetric models, we briefly revisit the conditions for enhanced supersymmetry. We also discuss some examples of non-abelian cosets with flux.

1 Introduction

Wess–Zumino–Witten (WZW) models and their cosets (gauged WZW models) provide examples of string backgrounds where both the exact CFT description and the geometry of the target space are well known. The coset space G/H is obtained by the identification $g \sim hgh^{-1}$ ($g \in G, h \in H$); hence its geometry is quite different from that of the usual left coset ($g \sim hg$). The "adjoint coset" is also required to have non-trivial dilaton and three-form flux ($H_{\mu\nu\lambda}$) on it in order to ensure conformal invariance.

For left cosets, the invariant one-forms and structure constants offer a clear intuitive picture of the geometry. In [1], analogous one-forms were introduced for adjoint cosets and were shown to define an orthonormal frame for the metric. The goal of this note is to take advantage of these one-forms to better understand the geometry of the adjoint coset with emphasis on the properties of the flux.

We first derive a simple, general expression for the flux in the orthonormal frame.[1] As a consistency check, we use it to verify conformal invariance to the leading order. We then specialize to supersymmetric cases and comment on the enhancement of world-sheet supersymmetry from $N = 1$ to $N = 2$ in the presence of the flux. Finally, we discuss the conditions for vanishing of the flux and two examples of non-abelian cosets with $\dim(G/H) = 6$. Our result may be useful in the study of how mirror symmetry works [3] (see also [4]) in an NS-NS flux background and the geometric aspects of D-branes in the gauged WZW model [5].

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[1] Throughout this paper, we work only in the semi-classical ($\alpha'/R^2 \sim 1/k \ll 1$) limit because the problem of obtaining the exact expression for the flux is quite involved [2].

2 Setup

We begin with a very brief review of the WZW model and its cosets to set up our notation. Let G be a compact, simple Lie group. The Lie algebra of G is written in terms of an orthonormal basis of anti-Hermitian generators as follows:

$$[T_A, T_B] = f_{AB}{}^C T_C, \qquad \mathrm{Tr}(T_A T_B) = -\delta_{AB}. \tag{1}$$

To describe the geometry of the group manifold, we introduce the standard one-forms:

$$\begin{aligned} &g^{-1}\mathrm{d}g = E^A T_A, \quad \mathrm{d}g\, g^{-1} = \tilde{E}^A T_A, \\ &\tilde{E}^A = C^{AB} E^B, \qquad C_{AB} = -\mathrm{Tr}(T_A g T_B g^{-1}), \\ &CC^T = 1. \end{aligned} \tag{2}$$

The WZW model defined for G,

$$S_G = -\frac{k}{4\pi} \int \mathrm{d}^2 z\, \mathrm{Tr}(g^{-1}\partial g \cdot g^{-1}\bar{\partial} g) + \mathrm{i}k\Gamma_{\mathrm{WZ}}, \tag{3}$$

corresponds to a sigma model on the group manifold with constant dilaton and the following metric and flux:

$$\mathrm{d}s^2 = E_A E_A, \quad H = \frac{1}{6} f_{ABC} E_A E_B E_C. \tag{4}$$

More precisely, the metric and the flux should be scaled by the radius square $R^2 = k\psi^2\alpha'/4$, where the integer k is the level of the WZW model and ψ is the highest root of Lie(G). We will suppress R^2 in the following unless its precise value becomes important.

We will consider cosets of type G/H, where $\mathrm{rank}(H) = \mathrm{rank}(G)$, and H acts on G as $g \to hgh^{-1}$. We use $(a, b, \ldots)$ indices for Lie(H) and $(\alpha, \beta, \ldots)$ indices for its orthogonal complement. The coset theory is realized as a gauged WZW theory with the following action and gauge transformation law:

$$S = S_G + S_A, \tag{5}$$

$$S_A = \frac{k}{2\pi}\int \mathrm{d}^2 z\,\mathrm{Tr}(\bar{A}g^{-1}\partial g - A\bar{\partial}gg^{-1} - \bar{A}A + g^{-1}Ag\bar{A})$$
$$= -\frac{k}{2\pi}\int \mathrm{d}^2 z(\bar{A}_a E^a - A_a\tilde{E}^a - A^a(\eta_{ab} - C_{ab})\bar{A}^b),$$
$$g \to u^{-1}gu, \quad A_i \to u^{-1}(A_i + \partial_i)u. \tag{6}$$

3 The expression

Since the action is quadratic in the non-propagating gauge field, it is easy to integrate out the gauge field and find [1,6]

$$G_{MN} = G^{(0)}_{MN} + 2(M^{-1})_{ab}E^a{}_{(M}\tilde{E}^b{}_{N)}, \tag{7}$$

$$B_{MN} = B^{(0)}_{MN} + 2(M^{-1})_{ab}E^a{}_{[M}\tilde{E}^b{}_{N]}, \tag{8}$$

$$e^{-2\phi} = \det\ M, \tag{9}$$

where $M_{ab} \equiv \delta_{ab} - C_{ab}$. Although G_{MN} and B_{MN} carry $d_G = \dim(G)$ indices, they actually depend only on the "coset directions," as can be seen from the existence of the $d_H = \dim(H)$ null vectors

$$Z_a{}^M = E_a{}^M - \tilde{E}_a{}^M = M_{ab}E_b{}^M - C_{a\beta}E_\beta{}^M$$
$$\implies \quad G_{MN}Z_a{}^M = 0. \tag{10}$$

The removal of d_H degrees of freedom in an gauge-invariant way can be made clear with the help of the one-forms [1]

$$H_\alpha = E_\alpha + E_a(M^{-1})_{ab}C_{b\alpha} \quad (Z_a \cdot H_\alpha = 0). \tag{11}$$

As shown in [1], these one-forms define an orthonormal frame, i.e.,

$$\mathrm{d}s^2 = H_\alpha H_\alpha. \tag{12}$$

It is natural to write down the flux also in this frame. A lengthy but straightforward computation using the basic identities

$$\mathrm{d}C_{AB} = -C_{AD}f_{DBC}E_C, \tag{13}$$

$$f_{ABC} = C_{AD}C_{BE}C_{CF}f_{DEF}, \tag{14}$$

$$f_{ACD}f_{BCD} = c_G\delta_{AB}, \quad f_{acd}f_{bcd} = c_H\delta_{ab}, \tag{15}$$

$$f_{AB[C}f^B{}_{DE]} = 0, \tag{16}$$

$$f_{ab\gamma} = 0, \tag{17}$$

shows that the flux also takes a very simple form in this frame:

$$H = \tfrac{1}{6}\left\{f_{\alpha\beta\gamma} + 3A_{[\alpha\beta\gamma]}\right\}H_\alpha \wedge H_\beta \wedge H_\gamma,$$
$$A_{\alpha\beta\gamma} = f_{a\alpha\beta}(M^{-1})_{ab}C_{b\gamma}. \tag{18}$$

This expression is the starting point of our discussion in what follows.

It is useful to note that the gauge transformation (6) translates into a local Lorentz transformation on the vielbeins H_α. Suppose we choose a gauge slice $g_0(x)$ with a set of coordinates $\{x^\mu\}$ ($\mu = 1, \ldots, d_G - d_H$). Then, consider the following type of gauge transformation:

$$g_0(x) \to h(f^m(x))\, g_0(x)\, h(f^m(x))^{-1}, \tag{19}$$

where $h(y^m)$ ($m = 1, \ldots, d_H$) define a coordinate system on H. The functions $f^m(x)$ shift the gauge slice from the original one without inducing a coordinate change. Upon this type of gauge transformation, the one-forms E_a, E_α and H_α transform as

$$E_a \to Q_{ab}(E_b - e_cM_{cb}),$$
$$E_\alpha \to Q_{\alpha\beta}(E_\beta + e_cC_{c\beta}), \tag{20}$$
$$H_a \to Q_{\alpha\beta}(x)H_\beta,$$

where $Q_{AB} = -\,\mathrm{Tr}(T_AhT_Bh^{-1})$ and $h^{-1}\mathrm{d}h = e_aT_a$.

Clearly, the change of gauge slice results in a local Lorentz transformation on H_α.

4 Conformal invariance

The leading order conformal invariance condition for a sigma model is well known to be

$$R_{MN} - \frac{1}{4}H_{MIJ}H_N{}^{IJ} + 2\nabla_M\nabla_N\phi = 0, \tag{21}$$

$$\nabla^M(e^{-2\phi}H_{MIJ}) = 0, \tag{22}$$

$$e^{2\phi}\nabla^2(e^{-2\phi}) - \frac{1}{6}H^2 = \Lambda. \tag{23}$$

For WZW or coset models, the constant Λ on the RHS of the third equations equals $2(\Delta d)/3\alpha'$, where (Δd) is the deviation of the "dimension of the target space" (more precisely, the central charge) from an integer value.

For a WZW model, it follows straightforwardly from $\mathrm{d}E_A = -\frac{1}{2}f_{ABC}E_B \wedge E_C$ that

$$4R_{AB} = H_{ACD}H_{BCD} = f_{ACD}f_{BCD} = c_G\delta_{AB}, \tag{24}$$

$$H^2 = \frac{c_Gd_G}{R^2} = \frac{4c_Gd_G}{k\psi^2\alpha'}. \tag{25}$$

At a large k, the value of H^2 agrees with the central charge of the WZW model at level k subtracted from its value in the $k \to \infty$ limit (recall $c = \frac{k\psi^2 d_G}{k\psi^2 + c_G}$). Equation (22) follows from the Jacobi identity for the structure constants.

For a coset space, the computation is somewhat more involved. As usual, the metric connection is derived from

$$\mathrm{d}H_\alpha = -\tfrac{1}{2}(f_{\alpha\beta\gamma} + A_{\beta\gamma\alpha})H_\beta \wedge H_\gamma$$
$$-(M^{-1})_{ab}f_{\alpha\beta b}H_\beta \wedge E_a. \tag{26}$$

The last term ensures that the spin connection $\omega_{\alpha\beta}$ transforms inhomogeneously under a local Lorentz transformation. It also produces many non-tensor terms in the intermediate steps of the computation of the curvature tensor. This complication can be avoided by using the gauge transformation (20) to set $E_a = 0$. This can be always done at any point on the coset space, although care should be taken to include the derivatives of E_a, which do not vanish in general. In this special gauge, the connection is given by

$$\begin{aligned}\omega_{\alpha\beta} &= -\tfrac{1}{2}(f_{\alpha\beta\gamma} - A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} - A_{\alpha\gamma\beta})H_\gamma \\ &\equiv \omega_{\alpha\beta\gamma}H_\gamma,\end{aligned} \tag{27}$$

and the components of its derivatives that are relevant in computing $R_{\alpha\beta}$ are

$$\begin{aligned}\mathrm{d}(\omega_{\alpha\beta\gamma}) &= \tfrac{1}{2}(A_{\alpha\beta\gamma|\delta} - A_{\beta\gamma\alpha|\delta} + A_{\alpha\gamma\beta|\delta})H_\delta \\ &\quad +\Delta\omega_{a\beta\gamma|\delta}H_\delta, \\ A_{\alpha\beta\gamma|\delta} &= A_{\alpha\beta\sigma}(A_{\sigma\delta\gamma} + f_{\sigma\delta\gamma}) + f_{a\beta\gamma}(M^{-1})_{ab}C_{bc}f_{c\delta\gamma}, \\ \Delta\omega_{\alpha\beta[\gamma|\delta]} &= -\tfrac{1}{2}(M^{-1})_{ab}f_{\alpha\beta b}f_{a\gamma\delta}.\end{aligned} \tag{28}$$

Using these results and the basic properties (13)– (17), it is straightforward to verify the conformal invariance conditions (24) including the precise value of Λ.

5 $N = 2$ supersymmetry

It is well known [7,8] that supersymmetry of the $N = 1$ G/H coset is enhanced to $N = 2$ when $\mathcal{T} \equiv \mathrm{Lie}(G) - \mathrm{Lie}(H)$ decomposes as $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$, where $\mathcal{T}_\pm$ are complex conjugate representations of H with $[\mathcal{T}_+, \mathcal{T}_+] \subset \mathcal{T}_+$, $[\mathcal{T}_-, \mathcal{T}_-] \subset \mathcal{T}_-$. In complex notation, closure under commutation implies that $f_{ijk} = 0 = f_{\bar{i}\bar{j}\bar{k}}$ and $f_{ija} = 0 = f_{\bar{i}\bar{j}a}$. It follows that the $(3,0)$ and $(0,3)$ components of the flux vanish. This fact is in agreement with a related analysis [9] of supersymmetry enhancement of sigma models in the presence of the flux; in [9], it was shown that in order for an $N = 1$ supersymmetric sigma model to have an extra supersymmetry, the target space should be complex and the $(3,0)$ and $(0,3)$ components of the flux should vanish.

6 Examples

Given the formula for the flux (18), it is natural to ask what the conditions are for a G/H coset to have non-vanishing flux. First, we note that the flux cannot vanish when $f_{\alpha\beta\gamma} \neq 0$. The reason is that $f_{\alpha\beta\gamma}$ and $A_{[\alpha\beta\gamma]}$ are orthogonal to each other ($f_{\alpha\beta\gamma}A_{\alpha\beta\gamma} = 0$) as follows from (15) and (17), and therefore cannot cancel each other. For $N = 2$ supersymmetric cosets (Kazama–Suzuki models), all such examples have been classified in [10]. The simplest among them is $SO(5)/SU(2) \times U(1)$ where $su(2)$ is embedded along a pair of long roots in $so(5)$.

For cosets with $f_{\alpha\beta\gamma} = 0$, it remains to determine when $A_{[\alpha\beta\gamma]}$ also vanishes. To our knowledge, the full answer to this question is not known. In the literature, all known examples with $f_{\alpha\beta\gamma} = 0$ and $A_{[\alpha\beta\gamma]} \neq 0$ are abelian cosets (i.e., the subset H is abelian) [11–15].[2] Several non-abelian cosets with $f_{\alpha\beta\gamma} = A_{[\alpha\beta\gamma]} = 0$ are also known [6, 17–23]. Using our formula (18) and a gauge choice similar to that of [6], we have computed the flux for the two Kazama–Suzuki models of dimension 6: $SU(4)/SU(3) \times U(1)$ and $SO(5)/SO(3) \times SO(2)$. It turns out that $A_{[\alpha\beta\gamma]}$ vanishes for the former and not for the latter. It would be interesting to develop a systematic method to determine whether a given coset with $f_{\alpha\beta\gamma} = 0$ has vanishing flux. An algebraic CFT description of coset models may turn out to be useful to proceed in that direction.

[2] See [16] for an example of a $(G \times G')/H$ coset that is rather different from the G/H cosets considered here.

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